



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

The Set of Eight Self-Associated Points in Space.

BY JOHN ROGERS MUSSELMAN.

Introduction.

Associated point sets were first discussed by Rosanes* and Sturm.† Later a type of self-conjugate association was treated by Study.‡ Recently Coble§ discussed the association of a set P_n^k (n -points in S_k) with a set Q_n^{n-k-2} (n -points in S_{n-k-2}) and derived the complete systems of invariants for P_6^1 and P_6^2 .

An interesting case of associated sets occurs when both sets of points are in the same space. The term association as defined implies a mutual ordering of the points. It may be possible to project the one set upon the other in the order of association. The two sets are then said to be self-associated, and the order will be referred to as the identical order. For the P_8^3 this is the well-known set of base points of a net of quadrics. If the associated sets can be projected one upon the other in some order other than the identical, the sets are said to be self-associated in other than the identical order. It is this type of P_8^3 which will be discussed in this paper. In § 2 the general set of eight points self-associated in some order is treated. If this self-association requires that the set be the base points of a net of quadrics, the set is said to be of type B and is discussed in § 3. If the eight points lie on a rational space cubic, the set is of type R and is considered in § 4. Various theorems and facts of immediate use are grouped in § 1. In order to restrict the number of cases, no order of self-association is discussed if in the set two points should coincide, three lie on a line, or four be in a plane.|| Such a set is said to be of type E .

Those sets of points which can be self-associated in other than the identical order, can naturally be determined only to within projective transformation.

* *Crelle*, Bd. LXXXVIII (1880), p. 241.

† *Math. Ann.*, Bd. I (1869), p. 533, and Bd. XXII (1883), p. 569.

‡ *Math. Ann.*, Bd. LX (1905), p. 321.

§ *Transactions*, Vol. XVI (1915), p. 155. This paper hereafter will be cited as C.

|| This excludes some cases of interest. See footnote on Desmic Systems, p. 81.

Most of the conclusions in § 2 are given in terms of the elliptic parameters of the points; if the curve on them degenerates, the geometrical construction of the set is given. In § 3 the ten types of the self-projective planar quartic given by Wiman* are tabulated and their connection, where possible, shown with specific orders of self-association of the P_8^3 . The types of self-associated sets on a rational cubic and the groups connected with them are given in § 4.

§ 1. Let the set of eight points in space be given by the equations

$$(up_1)=0, \quad (up_2)=0, \quad \dots, \quad (up_8)=0.$$

Since any five points in space are linearly related, these equations are connected by four linear relations. Let them be

$$q_{1i}(up_1) + q_{2i}(up_2) + \dots + q_{8i}(up_8) = 0, \quad i=1, 2, 3, 4.$$

Multiplying them respectively by v_i and adding, we have the single identity in u and v ,

$$(1) \quad (vq_1)(up_1) + (vq_2)(up_2) + \dots + (vq_8)(up_8) = 0,$$

which leads to the set Q_8^3 . Thus a set P_8^3 projectively defines an associated set Q_8^3 , and the relation is mutual. If the set P_8^3 is self-associated the above identity can be written as

$$(2) \quad \lambda_1(up_1)(vq_1) + \lambda_2(up_2)(vq_2) + \dots + \lambda_8(up_8)(vq_8) = 0,$$

where the points q_i are merely some permutation of the points p_i . The latter can be replaced by the two following identities:

$$(3) \quad \begin{aligned} \lambda_1(up_1)(uq_1) + \lambda_2(up_2)(uq_2) + \dots + \lambda_8(up_8)(uq_8) &= 0, \\ \lambda_1(p_1q_1x) + \lambda_2(p_2q_2x) + \dots + \lambda_8(p_8q_8x) &= 0. \end{aligned}$$

We shall have need of the following facts in the later paragraphs:

(4) If the associated sets P_8^3 and Q_8^3 in S_3 be placed so that the first five points of each constitute the same base, the remaining three points of each set lie in the same plane α , and form polar triangles of the conic Q_α , in α , which is apolar to all sections by α of the basic quadrics.†

(5) If by projection from one of the points q , say q_8 , and section by a plane, there is obtained from the remaining seven points $q_1, \dots, q_7$ a set Q_7^2 , this set is associated with the set P_7^3 formed by the points $p_1, \dots, p_7$ of P_8^3 .‡ From this theorem is obtained a result of importance for any order of association containing a cycle of two letters, say (p_7p_8) . By projection from p_8 we get seven points p in a plane associated with the set of seven points q . If the

* *Math. Ann.*, Bd. XLVIII (1897), p. 222.

† C., p. 159.

‡ C., p. 158.

sets P_8^3 and Q_8^3 are not self-associated, p_7 will have been sent into q_8 . Projecting now from q_8 we obtain six points q in a plane associated with six points p in the plane. If the sets P_8^3 and Q_8^3 are self-associated, then p_7 is q_8 , and the projected sets in the plane coincide and are self-associated. The set of six self-associated points in the plane has been fully treated by Coble,* and his conclusions are available for our purposes.

(6) Two sets associated with a third set are projective; for the sets are only projectively known so that a third set will projectively define its associated set.

There are twenty-one possible orders in which a P_8^3 may be self-associated, as (12), (12)(34), . . . , where the points are indicated by their subscripts and where, for example, by (12345) we mean $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$ is associated with $p_2, p_3, p_4, p_5, p_1, p_6, p_7, p_8$.

§ 2. In this section we treat those orders of self-association of the P_8^3 which permit the eight points to lie on a unique elliptic quartic. Three theorems are now proved which enable us to classify some of the orders of self-association as belonging to types *B* and *E*; such types are discussed later.

(7) *Any order of association containing only one cycle of three points is of type E.* Suppose the cycle to be (123) . . . , where the other five points enter in any possible arrangement save a cycle of three. By applying (6) this order of association (123) . . . implies projectivity in the order (312) . . . , and, consequently, is projective in that power of this order which is the least common multiple of the periods of the cycles—not including the cycle of three points. The original order of association is then projective to the order (123), let us say. The transformation defined by this projectivity has multipliers 1, ω , or ω^2 ; ($\omega^3=1$). Evidently two multipliers at least will be alike and we can choose these two to be unity. There are then three types to consider: 1, 1, 1, ω ; 1, 1, ω , ω ; 1, 1, ω , ω^2 . The transformation having the first set of multipliers has a fixed point and a fixed plane; the one with the second set of multipliers has two lines of fixed points, while the transformation with the third set of multipliers has a line of fixed points and a line of fixed planes. In all these types we can not have five fixed points without four of the P_8^3 lying in a plane and thus giving a P_8^3 of type *E*. Hence the following orders (123), (123)(45), (123)(45)(67), (123)(4567) and (123)(45678) are of type *E* and will not be discussed in this paper.

* C., p. 163.

(8) *Any order of association containing only one cycle of four points is of type E.* By an argument similar to that in (7) it can be shown that the order of association (1234) . . . implies projectivity to the order of association (13)(24). This type of projective transformation is the well-known harmonic perspectivity in a point and plane, or in two lines. In either case to have four fixed points would require four of the P_8^3 to lie in a plane; and so any order of association containing only one cycle of four points leads to a P_8^3 of type E. Hence the orders of association (1234), (1234)(56) and (1234)(56)(78) are of type E and will not be considered in this paper.

(9) *All further orders of association of odd period are of type B.* In fact all orders of association of odd period are of type B, but as some are included in the statements of the last two paragraphs, we shall say only those orders of odd period, not previously excluded, are of type B. Let Π be the order of association. By (6) the order of association Π implies projectivity to the order of association Π^2 . To be of type B, Π is projective to itself in the identical order, or $\Pi^{2n}=\Pi$. This says $\Pi^{2n-1}=1$ which is true if Π is of odd period. Hence all orders of association of odd period are of type B, and the following orders (123)(456), (12345) and (1234567) will be reserved for discussion in the next section of this paper.

(10) The remaining ten orders of self-association will now be taken up in detail, considering first the order (123456)(78). This order implies projectivity to the order (135)(246); thus defining a transformation with multipliers taken from $1, \omega, \omega^2$ ($\omega^3=1$). Let us give the P_8^3 the following coordinates:
 1: 1, 1, 1, 1, 2: λ , 1, x_3 , x_4 , 3: 1, 1, ω^2 , ω , 4: λ , 1, $\omega^2 x_3$, ωx_4 ,
 5: 1, 1, ω , ω^2 , 6: λ , 1, ωx_3 , $\omega^2 x_4$, 7: 1, 0, 0, 0, 8: 0, 1, 0, 0.
 The sixteen equations resulting from the identity

$$\lambda_1(up_1)(vp_2) + \dots + \lambda_6(up_6)(vp_1) + \lambda_7(up_7)(vp_8) + \lambda_8(up_8)(vp_7) \equiv 0$$

have the following matrix where the row $[ij]$ indicates the coefficients of the equation derived from the coefficient of $u_i v_j$ ($i, j=1, \dots, 4$).

[11]	λ	λ	λ	λ	λ	λ	0	0,
[22]	1	1	1	1	1	1	0	0,
[33]	x_3	$\omega^2 x_3$	ωx_3	x_3	$\omega^2 x_3$	ωx_3	0	0,
[44]	x_4	ωx_4	$\omega^2 x_4$	x_4	ωx_4	$\omega^2 x_4$	0	0,
[12]	1	λ	1	λ	1	λ	1	0,
[21]	λ	1	λ	1	λ	1	0	1,
[13]	x_3	$\lambda \omega^2$	$\omega^2 x_3$	$\lambda \omega$	ωx_3	λ	0	0,
[31]	λ	x_3	$\lambda \omega^2$	$\omega^2 x_3$	$\lambda \omega$	ωx_3	0	0,

[14]	x_4	$\lambda\omega$	ωx_4	$\lambda\omega^2$	$\omega^2 x_4$	λ	0	0,
[41]	λ	x_4	$\lambda\omega$	ωx_4	$\lambda\omega^2$	$\omega^2 x_4$	0	0,
[23]	x_3	ω^2	$\omega^2 x_3$	ω	ωx_3	1	0	0,
[32]	1	x_3	ω^2	$\omega^2 x_3$	ω	ωx_3	0	0,
[24]	x_4	ω	ωx_4	ω^2	$\omega^2 x_4$	1	0	0,
[42]	1	x_4	ω	ωx_4	ω^2	$\omega^2 x_4$	0	0,
[34]	x_4	ωx_3	x_4	ωx_3	x_4	ωx_3	0	0,
[43]	x_3	$\omega^2 x_4$	x_3	$\omega^2 x_4$	x_3	$\omega^2 x_4$	0	0.

We can use [12] and [21] to determine λ_7 and λ_8 after the other equations have been satisfied. Both [33] and [44] can be divided respectively by x_3 and x_4 (which can not be zero else four points in a plane); using them in connection with [22] to eliminate $\lambda_1, \lambda_3, \lambda_5$ from the other equations we get the following five equations in $\lambda_2, \lambda_4, \lambda_6$:

$$\begin{aligned} (x_3 - \omega\lambda)(\lambda_2 + \omega^2\lambda_4 + \omega\lambda_6) &= 0, & (x_4 - \omega^2)(\lambda_2 + \omega\lambda_4 + \omega^2\lambda_6) &= 0, \\ (x_3 - \omega)(\lambda_2 + \omega^2\lambda_4 + \omega\lambda_6) &= 0, & (\omega x_3 - x_4)(\lambda_2 + \lambda_4 + \lambda_6) &= 0, \\ (x_4 - \omega^2\lambda)(\lambda_2 + \omega\lambda_4 + \omega^2\lambda_6) &= 0, \end{aligned}$$

If $\lambda_2 + \lambda_4 + \lambda_6 = 0$, and $\lambda_2 + \omega^2\lambda_4 + \omega\lambda_6 = 0$, then $\lambda_2 + \omega\lambda_4 + \omega^2\lambda_6 \neq 0$, $x_4 = \omega^2$. $\lambda = 1$.

If $\lambda_2 + \lambda_4 + \lambda_6 = 0$, and if $\lambda_2 + \omega\lambda_4 + \omega^2\lambda_6 = 0$, then $\lambda_2 + \omega^2\lambda_4 + \omega\lambda_6 \neq 0$, and $x_3 = \omega$ and $\lambda = 1$.

But $\lambda \neq 1$, else six points of the P_8^3 are in a plane whence

$$\begin{aligned} \lambda_2 + \omega\lambda_4 + \omega^2\lambda_6 &= 0, & \lambda_2 + \omega^2\lambda_4 + \omega\lambda_6 &= 0, \\ \lambda_2 + \lambda_4 + \lambda_6 &\neq 0, & \text{and} & & x_4 &= \omega x_3. \end{aligned}$$

With this relation on the constants, the set P_8^3 has the following coordinates:

$$\begin{array}{llll} 1: 1, 1, 1, 1, & 2: a, 1, b, \omega b, & 3: 1, 1, \omega^2, \omega, & 4: a, 1, \omega^2 b, \omega^2 b, \\ 5: 1, 1, \omega, \omega^2, & 6: a, 1, \omega b, b, & 7: 1, 0, 0, 0, & 8: 0, 1, 0, 0. \end{array}$$

But points 7, 8, 1, 4; 7, 8, 2, 5; 7, 8, 3, 6 lie in planes so the order of association (123456) (78) leads to a P_8^3 of type E and will not be discussed further.

(11) The application of theorem (4) shows that the order of self-association (12) is possible. Choosing 4, 5, 6, 7, 8 as the base points we have to require that $\overline{13}$ and $\overline{23}$ be tangent lines of the conic Q_a . Analytically we see that, from the two identities,

$$\begin{aligned} (\lambda_1 + \lambda_2)(up_1)(up_2) + \lambda_3(up_3)^2 + \dots + \lambda_8(up_8)^2 &= 0, \\ (\lambda_1 - \lambda_2)(p_1 p_2 x) &= 0, \end{aligned}$$

unless two points coincide, $\lambda_1 = \lambda_2$. Quadrics on the six points 3, 4, ..., 8 must cut the line $\overline{12}$ harmonically. There are four independent quadrics on

six points, so 1 and 2 must be apolar to these quadrics and are therefore corresponding points of the Weddle surface determined by the other six points. The quartic curve on the P_8^3 has degenerated, for 1 and 2 lie on a bisecant of the cubic curve through the other six points. This set contains five absolute constants.

(12) The second identity for the order of association (12) (34) requires $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$ unless two points coincide, or four lie on a line. Using points 5, 6, 7, 8 as reference points, 4 as unit point and 1, 2, 3 as x, y, z , respectively, the ten equations resulting from the first identity are:

$$\begin{aligned} 2\lambda_1 x_1 y_1 + 2\lambda_3 z_1 + \lambda_5 &= 0, & \lambda_1 (x_1 y_2 + x_2 y_1) + \lambda_3 (z_1 + z_2) &= 0, \\ 2\lambda_1 x_2 y_2 + 2\lambda_3 z_2 + \lambda_6 &= 0, & \lambda_1 (x_1 y_3 + x_3 y_1) + \lambda_3 (z_1 + z_3) &= 0, \\ 2\lambda_1 x_3 y_3 + 2\lambda_3 z_3 + \lambda_7 &= 0, & \lambda_1 (x_1 y_4 + x_4 y_1) + \lambda_3 (z_1 + z_4) &= 0, \\ 2\lambda_1 x_4 y_4 + 2\lambda_3 z_4 + \lambda_8 &= 0, \\ & \lambda_1 (x_2 y_3 + x_3 y_2) + \lambda_3 (z_2 + z_3) &= 0, \\ & \lambda_1 (x_2 y_4 + x_4 y_2) + \lambda_3 (z_2 + z_4) &= 0, \\ & \lambda_1 (x_3 y_4 + x_4 y_3) + \lambda_3 (z_3 + z_4) &= 0. \end{aligned}$$

Eliminating $\lambda_1, \lambda_3 z_1, \lambda_3 z_2, -\lambda_3 z_4$ we obtain:

$$\begin{aligned} x_1 (y_3 - y_4) + x_2 (y_4 - y_3) + x_3 (y_1 - y_2) + x_4 (y_2 - y_1) &= 0, \\ x_1 (y_2 - y_3) + x_2 (y_1 - y_4) + x_3 (y_4 - y_1) + x_4 (y_3 - y_2) &= 0, \\ x_1 (y_4 - y_2) + x_2 (y_3 - y_1) + x_3 (y_2 - y_4) + x_4 (y_1 - y_3) &= 0. \end{aligned}$$

If these three equations can be satisfied, the ten equations above can likewise be satisfied. These three equations represent for given y a pencil of planes, so x runs along a line. From their symmetry for given x, y is on a line. Hence this P_8^3 will contain four absolute constants.

The relations connecting x and z are:

$$\begin{aligned} x_2 x_4 (z_1 + z_3) - x_2 x_3 (z_1 + z_4) - x_1 x_4 (z_2 + z_3) + x_1 x_3 (z_2 + z_4) &= 0, \quad A \\ x_2 x_4 (z_1 + z_3) - x_3 x_4 (z_1 + z_2) - x_1 x_2 (z_3 + z_4) + x_1 x_3 (z_2 + z_4) &= 0, \quad B \\ x_1 x_2 (z_3 + z_4) - x_1 x_4 (z_2 + z_3) - x_2 x_3 (z_1 + z_4) + x_3 x_4 (z_1 + z_2) &= 0, \quad C \end{aligned}$$

three of the pencil of quadrics containing the quartic on the eight points. These surfaces can easily be constructed for A has for generators the lines $\overline{56}, \overline{78}$, and the line joining 3 to the fourth harmonic of 4 as to $\overline{56}, \overline{78}$. Similarly, B has for generators $\overline{58}, \overline{67}$, and the line joining 3 to the fourth harmonic of 4 as to $\overline{58}, \overline{67}$; while the generators of C are the lines $\overline{57}, \overline{68}$, and the join of 3 to the fourth harmonic of 4 as to $\overline{57}, \overline{68}$. The quadrics are cut in line pairs by the planes of the reference tetrahedron, since its edges are their generators. Thus the plane $x_1 = 0$ cuts the quadrics respectively in $x_2[x_4(z_1 + z_3) - x_3(z_1 + z_4)]$, $x_4[x_2(z_1 + z_3) - x_3(z_1 + z_2)]$ and $x_3[x_4(z_1 + z_2) - x_2(z_1 + z_4)]$. Three of these lines

are the sides of the reference triangle in the plane $x_1=0$, and the other three are lines through the vertices of the triangle meeting in a point. Moreover, this point is on the quartic curve. For any plane cuts the pencil of quadrics in a pencil of conics on four points of the curve. The conics here are line pairs on the vertices of the reference triangle. They must have another point in common which can not lie on the sides of the triangle, hence the lines through the vertices meet in a point on the curve. We thus obtain four more points on the curve, namely:

$$\begin{array}{ccccccccc} 0, & z_1+z_2, & z_1+z_3, & z_1+z_4, & \alpha & z_3+z_1, & z_2+z_2, & 0, & z_3+z_4, & \gamma \\ z_2+z_1, & 0, & z_2+z_3, & z_2+z_4, & \beta & z_4+z_1, & z_4+z_2, & z_4+z_3, & 0. & \delta \end{array}$$

But 3, 4, α , 5 are in a plane; so also 3, 4, β , 6; 3, 4, γ , 7 and 3, 4, δ , 8.

Calling the elliptic parameters of the points 1, 2, ..., 8, $u_1, u_2, \dots, u_8$, since 6, 7, 8, α are in a plane,

$$u_3+u_4+u_5 \equiv u_6+u_7+u_8, \text{ or } u_3+u_4 \equiv -u_5+u_6+u_7+u_8.$$

Similarly, $u_3+u_4 \equiv u_5-u_6+u_7+u_8 \equiv u_5+u_6-u_7+u_8 \equiv u_5+u_6+u_7-u_8$,

whence $2(u_3+u_4) = 4u_5$, or $-u_3-u_4+2u_5=0$.

Also $-u_3-u_4+2u_6=0$, $-u_3-u_4+2u_7=0$, $-u_3-u_4+2u_8=0$.

These relations say that the four planes on $\overline{-u_3}, \overline{-u_4}$, tangent to the curve, are tangents at the points u_5, u_6, u_7, u_8 . Find that quadric of the pencil having $\overline{u_3}, \overline{u_4}$ as generator; a plane through this line will cut the quadric in $\overline{-u_3}, \overline{-u_4}$. The four planes on this line $\overline{-u_3}, \overline{-u_4}$ tangent to the curve cut out the points u_5, u_6, u_7, u_8 . Points 1 and 2 are treated like 3 and 4. Hence $\overline{12}, \overline{34}$ are any two generators of the same system of a quadric on the curve; 5, 6, 7, 8 are the points where the curve is tangent to the generators of this same system. The P_3^2 thus contains four absolute constants, and the conditions on the parameters of the points are

$$u_1+u_2=u_3+u_4=2u_5=2u_6=2u_7=2u_8.$$

(13) The second identity for the order of association (12) (34) (56) (78) is

$$(\lambda_1-\lambda_2)(12x) + (\lambda_3-\lambda_4)(34x) + \dots + (\lambda_7-\lambda_8)(78x) \equiv 0.$$

If one difference vanishes, say the first, the lines $\overline{34}, \overline{56}$, and $\overline{78}$ are on a point, whence four points lie in a plane. If two differences vanish, either two pairs of points coincide or four points lie on a line. If three differences vanish, two points must coincide. These conditions all lead to sets of type *E*. If none of the differences vanish, the four lines are lines of the same system of generators on a quadric. Moreover, this condition is sufficient because the first

identity can be satisfied by choosing $\lambda_1 + \lambda_2 = 0, \dots, \dots, \lambda_7 + \lambda_8 = 0$. This set involves six absolute constants: one for the curve, one for the quadric, and one for each generator. In terms of the elliptic parameters of the points we have

$$u_1 + u_2 = u_3 + u_4 = u_5 + u_6 = u_7 + u_8 = k.$$

However the case remains for which all the differences may vanish. Applying theorem (5) and projecting from point 7 (or 8) we get six self-associated points in a plane which under the order of association (12) (34) (56) requires the points to lie on three lines meeting in a point.* This means that looking from point 7 (or 8) the projections of the lines $\overline{12}, \overline{34}, \overline{56}$ upon a plane meet in a point. To do so the line of perspection from 7 (or 8) must meet these lines. The lines of perspection from 7 and 8 are distinct, else four points in a plane. So 7 and 8 lie on two cross generators on the quadric having $\overline{12}, \overline{34}, \overline{56}$ as generators. Either the four lines are generators of the same system on a quadric, or the quadrics having $\overline{12}, \overline{34}, \overline{56}$ and $\overline{34}, \overline{56}, \overline{78}$, respectively, as generators are distinct. Since both are quadrics on the curve, their intersection, $\overline{34}, \overline{56}$ and two cross generators, shows 1, 2, 7, 8 must lie on the cross generators whence four points would be in a plane. Hence the two quadrics must be the same one and $\overline{12}, \overline{34}, \overline{56}, \overline{78}$ are generators of the same system on a quadric.

The first identity, $\lambda_1(u_1)(u_2) + \dots + \dots + \lambda_7(u_7)(u_8) = 0$, says that any quadric apolar to the first three pairs must cut $\overline{78}$ harmonically as to 7 and 8. Examine the quadric made of planes $\overline{134}, \overline{156}$. The plane $\overline{134}$, being tangent along the generator $\overline{34}$, cuts out on the quadric that generator of the other system through the point 1. But the plane $\overline{156}$ cuts out the same generator. Now the points where these lines cut $\overline{78}$, since the generators coincide, must be at 7 or at 8. In either case, four points lie in a plane, and we are led to a P_3^3 of type *E*. Therefore the order (12) (34) (56) (78) requires the four lines $\overline{12}, \overline{34}, \overline{56}, \overline{78}$ to be lines of the same system on a quadric. The set contains six constants and is given parametrically by

$$u_1 + u_2 = u_3 + u_4 = u_5 + u_6 = u_7 + u_8 = k.$$

(14) We can dispose of the order of association (123) (456) (78) in a few words. It implies projectivity to the order (132) (465). It is also associated in the cube of its order, that is, in the order (78). These facts enable us to construct the set. The six points 1, 2, 3, 4, 5, 6 determine a cubic curve on which they lie in two cyclic sets 1, 2, 3 and 4, 5, 6. The transformation sending

* C., p. 161.

these points into each other cyclically has two fixed points on the curve. Join these points, and 7 and 8 are points on the Weddle, determined by the first six points, cut out by this line of fixed points. The quartic curve has degenerated into a cubic and its bisecant, while the P_8^3 contains but one absolute constant.

(15) The order of association (1234)(5678) implies projectivity to the order (13)(24)(57)(68). The points must lie harmonically separated in pairs as to two fixed lines. Let them have the coordinates:

$$\begin{array}{llll} 1: 1, 0, 0, a, & 2: 0, 1, b, 0, & 3: 1, 0, 0, -a, & 4: 0, 1, -b, 0, \\ 5: 1, 1, 1, 1, & 6: x_1, x_2, x_3, x_4, & 7: 1, 1, -1, -1, & 8: x_1, x_2, -x_3, -x_4. \end{array}$$

Substituting these values in the identity we obtain sixteen equations which can be satisfied if the following conditions are satisfied:

$$x_4^2 + x_1^2 = x_3^2 + x_2^2 = x_1x_2 + x_3x_4 = x_1x_3 - x_2x_4 = 0, \quad (ax_3 - bx_2)^2 + (ax_1 + bx_4)^2 = 0.$$

The first four relations are satisfied by $x_1 = ix_4, x_3 = -ix_2 [i^4 = 1]$. Putting these values in the remaining condition,

$$(-iax_2 - bx_2)^2 + (aix_4 + bx_4)^2 = 0, \quad \text{or} \quad (x_2^2 + x_4^2)(ia + b)^2 = 0,$$

whence either $x_2^2 + x_4^2 = 0$, or $a = ib$.

CASE I. $x_1 = ix_4, x_3 = -ix_2$ and $x_2 = \pm ix_4$.

If $x_2 = ix_4$, letting $x_4 = 1$, we have $x_1 = i, x_2 = i, x_3 = 1, x_4 = 1$. With these values points 5, 6, 7, 8 are in a plane, so they lead to a P_8^3 of type E .

If $x_2 = -ix_4$, letting $x_4 = 1$, we have $x_1 = i, x_2 = -i, x_3 = -1, x_4 = 1$, which gives a P_8^3 involving two constants, and whose coordinates are:

$$\begin{array}{llll} 1: 1, 0, 0, a, & 2: 0, 1, b, 0, & 3: 1, 0, 0, -a, & 4: 0, 1, b, 0, \\ 5: 1, 1, 1, 1, & 6: 1, -1, i, -i, & 7: 1, 1, -1, -1, & 8: 1, -1, -i, i. \end{array}$$

CASE II. $x_1 = ix_4, x_3 = -ix_2, a = ib$ gives a P_8^3 containing two absolute constants, whose coordinates are:

$$\begin{array}{llll} 1: 1, 0, 0, ib, & 2: 0, 1, b, 0, & 3: 1, 0, 0, -ib, & 4: 0, 1, -b, 0, \\ 5: 1, 1, 1, 1, & 6: 1, -ic, -c, -i, & 7: 1, 1, -1, -1, & 8: 1, -ic, c, i. \end{array}$$

Both P_8^3 s give the lines $\overline{13}, \overline{24}, \overline{57}, \overline{68}$ as generators of the same system of a quadric on the curve.

In Case I the four generators are harmonic, and the double-ratio of the generators through 1, 2, 5, 6 on the edge $\overline{e_1e_4}$ of the reference tetrahedron is

$$\frac{(a-1)(b+i)}{(b-1)(a+i)}.$$

In Case II the double-ratio of $\overline{13}, \overline{24}, \overline{57}, \overline{68}$ is ic , and the double-ratio of the generators, on $\overline{e_1e_4}$, through the points 1, 2, 5, 6 is $\frac{(b+i)^2}{b^2-1}$.

(16) The order of association (12345)(67) implies projectivity to the order (13524). Let P_8^3 be

$$\begin{array}{llll} 1: 1, & 1, & 1, & 1, \\ 5: 1, & \epsilon^{2a}, & \epsilon^{2b}, & \epsilon^{2c}, \end{array} \quad \begin{array}{llll} 2: 1, & \epsilon^{3a}, & \epsilon^{3b}, & \epsilon^{3c}, \\ 6: 1, & 0, & 0, & 0, \end{array} \quad \begin{array}{llll} 3: 1, & \epsilon^a, & \epsilon^b, & \epsilon^c, \\ 7: 0, & 1, & 0, & 0, \end{array} \quad \begin{array}{llll} 4: 1, & \epsilon^{4a}, & \epsilon^{4b}, & \epsilon^{4c}, \\ 8: 0, & 0, & 1, & 0, \end{array}$$

where a, b, c are all different and different from zero, and ϵ^a is a fifth root of unity.

The identity (2) for the order of association (12345)(67) is:

$$\lambda_1(u_1)(v_2) + \dots + \lambda_5(u_5)(v_1) + \lambda_6(u_6)(v_7) + \dots + \lambda_8(u_8)(v_8) \equiv 0.$$

The coefficients of u_1v_2, u_2v_1, u_3v_3 determine $\lambda_6, \lambda_7, \lambda_8$, respectively. The remaining coefficients contain only the first five λ 's and reduce to eight equations whose matrix is:

$$\begin{array}{ccccc} 1 & \epsilon^a & \epsilon^{2a} & \epsilon^{3a} & \epsilon^{4a}, \\ 1 & \epsilon^c & \epsilon^{2c} & \epsilon^{3c} & \epsilon^{4c}, \\ 1 & 1 & 1 & 1 & 1, \\ 1 & \epsilon^{3b} & \epsilon^b & \epsilon^{4b} & \epsilon^{2b}, \\ 1 & \epsilon^{3c} & \epsilon^c & \epsilon^{4c} & \epsilon^{2c}, \\ 1 & \epsilon^{3(a+b)} & \epsilon^{(a+b)} & \epsilon^{4(a+b)} & \epsilon^{2(a+b)}, \\ 1 & \epsilon^{3(a+c)} & \epsilon^{(a+c)} & \epsilon^{4(a+c)} & \epsilon^{2(a+c)}, \\ 1 & \epsilon^{3(b+c)} & \epsilon^{(b+c)} & \epsilon^{4(b+c)} & \epsilon^{2(b+c)}. \end{array}$$

Let ϵ^d be the remaining fifth root of unity not used in the coordinates of the points. We have two possibilities.

$$\begin{array}{ll} \text{CASE I. } a+b=5, & c+d=5, \\ \text{I}\alpha. a=1, & b=4, c=2, d=3, \\ \text{I}\beta. a=1, & b=4, c=3, d=2. \end{array} \quad \begin{array}{ll} \text{CASE II. } a+c=5, & b+d=5, \\ \text{II}\alpha. a=1, & b=2, c=4, d=3, \\ \text{II}\beta. a=1, & b=3, c=4, d=2. \end{array}$$

We can always choose $a=1$. If it is not 1, by a proper power of the transformation we can make it 1. Using the above values for a, b, c, d we find the set of eight equations in $\lambda_1, \dots, \lambda_5$ incapable of solution except for the values in II β . Hence the P_8^3 contains no absolute constant and has the following coordinates:

$$\begin{array}{llll} 1: 1, & 1, & 1, & 1, \\ 5: 1, & \epsilon^2, & \epsilon, & \epsilon^3, \end{array} \quad \begin{array}{llll} 2: 1, & \epsilon^3, & \epsilon^4, & \epsilon^2, \\ 6: 1, & 0, & 0, & 0, \end{array} \quad \begin{array}{llll} 3: 1, & \epsilon, & \epsilon^3, & \epsilon^4, \\ 7: 0, & 1, & 0, & 0, \end{array} \quad \begin{array}{llll} 4: 1, & \epsilon^4, & \epsilon^2, & \epsilon, \\ 8: 0, & 0, & 1, & 0. \end{array}$$

(17) The order of association (123456) implies projectivity to the order (135)(246). Let the P_8^3 be

$$\begin{array}{llll} 1: 1, & 1, & 1, & 1, \\ 5: 1, & 1, & \omega, & \omega^2, \end{array} \quad \begin{array}{llll} 2: \lambda, & 1, & x_3, & x_4, \\ 6: \lambda, & 1, & \omega x_3, & \omega^2 x_4, \end{array} \quad \begin{array}{llll} 3: 1, & 1, & \omega^2, & \omega, \\ 7: 1, & 0, & 0, & 0, \end{array} \quad \begin{array}{llll} 4: \lambda, & 1, & \omega^2 x_3, & \omega x_4, \\ 8: 0, & 1, & 0, & 0. \end{array}$$

The sixteen equations resulting from the identity are satisfied, for the above choice of coordinates, if $\lambda = -1$, and $x_4 = -\omega x_3$. Hence the set contains one absolute constant and can be written as

$$\begin{array}{llll} 1: 1, 1, 1, & 2: -1, 1, & \alpha, -\omega\alpha, & 3: 1, 1, \omega^2, \omega, & 4: -1, 1, \omega^2\alpha, -\omega^2\alpha, \\ 5: 1, 1, \omega, \omega^2, & 6: -1, 1, \omega\alpha, -\alpha, & 7: 1, 0, 0, 0, & 8: 0, 1, 0, 0. \end{array}$$

To construct the set choose 7, 8, H_1, H_2 as the reference points, and 1 as unit point. Let H_3 be the point on $\overline{78}$ cut out by the plane $\overline{1H_1H_2}$ and P be the fourth harmonic on $\overline{78}$ of H_3 as to 7 and 8. Then 3 and 5 are determined as those points which with 1 have $H_1H_2H_3$ as their Hessian triangle. Select 4' in the plane $\overline{H_1H_2H_3}$ as any point on the harmonic line of $\overline{H_31}$ as to $\overline{H_3H_1}$ and $\overline{H_3H_2}$; likewise determining 5' and 6' as those points which with 4' have $H_1H_2H_3$ as their Hessian triangle. Projecting from 7 upon $\overline{PH_1H_2}$ the points 4', 5', 6' project into 4, 5, 6, whence the set is determined completely. By changing the sign of α , we get a set 4'', 5'', 6'' in $\overline{H_1H_2H_3}$, which projected from 8 gives 4, 5, 6 in the plane $\overline{PH_1H_2}$. The set of points lie on a cubic through 1, 2, ..., 6 and a bisecant on which lie 7 and 8. Hence the order of association (123456) is possible and the set of points contains one absolute constant.

(18) The order of association (12345678) implies projectivity to the order (1357) (2468). Let the P_8^3 be

$$\begin{array}{llll} 1: 1^1, 1, & 1, & 1, & 2: x_1, x_2, x_3, x_4, & 3: 1, i, -1, -i, \\ 4: x_1, ix_2, -x_3, -ix_4, & 5: 1, -1, 1, -1, & 6: x_1, -x_2, x_3, -x_4, \\ & 7: 1, -i, -1, i, & 8: x_1, -ix_2, -x_3, ix_4. \end{array}$$

The sixteen equations resulting from substituting these values in the identity are satisfied if we choose x_1, x_2, x_3, x_4 to satisfy $i(x_1^2 - x_3^2) - (x_2^2 - x_4^2) = 0$, whence the set contains two absolute constants. The quadric passes through 2 and 6, and has $\overline{13}, \overline{57}, \overline{15}, \overline{37}$ as generators. This apparent lack of symmetry is explainable. In the order (12345678) we pick out the points 2 and 6, the numbers enclosing them are 1, 3, 5, 7, hence we use for generators besides $\overline{15}, \overline{37}$, the lines $\overline{13}, \overline{57}$. Similarly, if we isolate points 4 and 8, the numbers enclosing them are 3, 5, 1, 7; hence there is a quadric having $\overline{15}, \overline{37}, \overline{35}, \overline{17}$ as generators, and on 4 and 8, namely:

$$i(x_1^2 - x_3^2) + (x_2^2 - x_4^2) = 0.$$

If we choose $x_4 = 1, x_3 = b, x_2 = a$, then $x_1 = \sqrt{b^2 + i(a^2 - 1)}$. Hence the order of association (12345678) is possible, the set contains two absolute constants, lies on an elliptic quartic, and its coordinates are given above.

(19) For the order of association (12) (34) (56) the identities are satisfied if we take points 5, 6, 7, 8 as the reference points, 4 as unit point, and 1, 2, 3 as x, y, z , respectively, by $\lambda_1=\lambda_2, \lambda_3=\lambda_4, \lambda_5=\lambda_6$, and if the following equations are satisfied,

$$\begin{aligned}x_2y_3 + x_3y_2 + x_1y_1 - x_2y_2 - x_1y_3 - x_3y_1 &= 0, \\x_2y_4 + x_4y_2 + x_1y_1 - x_2y_2 - x_1y_4 + x_4y_1 &= 0, \\x_3y_4 + x_4y_3 + 2x_1y_1 - x_1y_3 - x_3y_1 - x_1y_4 - x_4y_1 &= 0.\end{aligned}$$

Solving for y in terms of x we note:

$$y_1=y_2=y_3=y_4=x_1-x_2[2x_3x_4+2x_1x_2-x_2x_4-x_2x_3-x_1x_3-x_1x_4].$$

Now the four values of y_i can not be equal, else this point would coincide with 4, neither can $x_1=x_2$, or four points would be in a plane, whence point 1 must lie on the above quadric, and then point 2 will lie on a line. Point 3 is determined by the conditions $z_1:z_2:z_3:z_4=x_1y_1:x_2y_2:x_1y_3+x_3y_1-x_1y_1:x_1y_4+x_4y_1-x_1y_1$. Hence this P_8^3 contains three absolute constants. If we choose $x_1=1, x_2=a, x_3=b$, then $x_4=\frac{b+ab-2a}{2b-a-1}$.

The coordinates of point 2 are

$$y_1=1+\lambda, \quad y_2=\frac{b-1}{b-a}(1-\lambda), \quad y_3=\frac{2\lambda(b-1)}{a-1}, \quad y_4=\frac{2(b-1)}{2b-a-1};$$

while point 3 is

$$\begin{aligned}z_1=1+\lambda, \quad z_2=\frac{a}{b-a}[(b-1)(1-\lambda)], \quad z_3=\frac{b-1}{a-1}[2\lambda+(1+\lambda)(a-1)], \\z_4=\frac{(b-1)[2+(1+\lambda)(a-1)]}{2b-a-1}.\end{aligned}$$

Therefore the order of association (12) (34) (56) is possible; the set contains three absolute constants and its coordinates are given above.

§ 3. In this section we shall treat the P_8^3 when they are the base points of a net of quadrics. As such they are self-associated in the identical order, and if self-associated in some given order they are therefore projective in that order. The connection between the general planar quartic (genus 3) and the net of cubic curves on seven points in the plane is well known. The ∞^2 elliptic quartics on the eight base points project from one of those points into a net of cubics on seven points in the plane. Hence an intimate relation exists between the eight base points and the planar quartic. If now the P_8^3 is self-associated in some order and thereby projective in that order, we get a birational transformation of the planar quartic into itself, which is, moreover, a collineation. The types of self-projective quartics have been tabulated by Wiman, and in

this section we shall, where possible, connect each type with an order of self-association. Since the planar quartic is not of genus 3, if two points coincide, three lie on a line, or four be in a plane,* no orders of self-association will enter here which were of type *E* and which were not discussed in the preceding section.

Associated with the planar quartic are its thirty-six systems of contact cubics. If a collineation permutes some of these systems leaving at least one system fixed, there will be a self-association corresponding to it of the P_8^3 . If, however, all of these thirty-six systems are permuted under the collineation, then we can not connect an order of self-association with the quartic. The twenty-eight double-tangents of the quartic shall be designated by the twenty-eight symbols [12], ..., [78], thus furnishing at a glance the number of double-tangents fixed under an order of association.

The ten types of self-projective quartics are:

$$\begin{array}{ll}
 1^\circ & x_3^4 + x_3^2 f_2(x_1, x_2) + f_4(x_1, x_2) = 0, \\
 2^\circ & x_3^3 f_1(x_1, x_2) + f_4(x_1, x_2) = 0, \\
 3^\circ & ax_3^2 x_2^2 + bx_3 x_2 x_1^2 + x_3^3 x_1 + x_2^3 x_1 + x_1^4 = 0, \dagger \\
 4^\circ & x_3^4 + f_4(x_1, x_2) = 0, \\
 5^\circ & x_3^4 + ax_3^2 x_1 x_2 + x_1^4 + bx_1^2 x_2^2 + x_2^4 = 0, \dagger \\
 6^\circ & x_3^3 x_1 + x_1^4 + x_1^2 x_2^2 + x_2^4 = 0, \\
 7^\circ & x_3^3 x_2 + x_2^3 x_1 + x_1^3 x_3 = 0, \\
 8^\circ & x_3^4 + x_1^3 x_2 + x_1 x_2^3 = 0, \\
 9^\circ & x_3^3 x_1 + x_1^3 x_2 + x_2^4 = 0, \\
 10^\circ & x_3^4 + x_2^4 + x_1 x_2^3 = 0.
 \end{array}$$

Quartic 1° has four fixed double-tangents. This excludes the orders of association (12) and (12) (34) as they both have more fixed lines; leaving the orders (12) (34) (56) and (12) (34) (56) (78) to be considered. Apply theorem (5) to the order (12) (34) (56). The order of association (12) (34) on the six projected points in a plane requires (a) the six points to lie on a conic, or (b) four on a line, or (c) six on a line.‡ The last two conditions would require the P_8^3 to have four points at least in a plane, while (a) makes the points lie on a quadric cone. If this happens the planar quartic associated with the P_8^3 has a double-point and is no longer of genus 3. Hence, if we can connect any order of association with quartic 1° , it must be the order (12) (34) (56) (78). This we can do, and the conditions on the elliptic parameters of the points are $u_1 + u_2 = u_3 + u_4 = u_5 + u_6 = u_7 + u_8 = p/4$ (p = period). This set contains four absolute constants; the modulus is not an absolute constant, and this necessarily is the number of constants in the equation 1° .

* This excludes the P_8^3 which form two desmic tetrahedra, which set is unaltered by a G_{192} . Of course this P_8^3 is of type *B*, and the planar quartic associated with it is the complete quadrilateral.

† Types given by Wiman are $x_3^2 x_2^2 + x_3 x_2 x_1^2 + x_3^3 x_1 + x_2^3 x_1 + x_1^4 = 0$, and $x_3^4 + x_3^2 x_1 x_2 + x_1^4 + x_1^2 x_2^2 + x_2^4 = 0$. The most general quartics of these types contain two constants, and are as given above.

‡ C., p. 161.

Quartic 2° is invariant under a perspective G_3 and has $x_3=0$ as a fixed undulation tangent. The group associated with quartic 3° is a non-perspective G_3 with $x_1=0$ as a fixed double-tangent. Of the two orders of association of period 3, we discard (123), since it has too many fixed lines, and study (123)(456). The P_8^3 is associated and projective in the order (123)(456). The identity can be satisfied if the set has the following coordinates:

$$\begin{array}{llll} 1: 1, 1, 1, 1, & 2: 1, 1, \omega^2, \omega, & 3: 1, 1, \omega, \omega^2, & 4: ab, 1, a, b, \\ 5: ab, 1, \omega^2a, \omega b, & 6: ab, 1, \omega a, \omega^2b, & 7: 1, 0, 0, 0, & 8: 0, 1, 0, 0. \end{array}$$

The set contains two absolute constants, and lies on a cubic curve and its bisecant. Given the curve and points 1, 2, 3 on it, they determine a cyclic transformation of period 3. Choose 4 as any point on the curve, then 5 and 6 are determined as those points which with 4 form a cyclic set under the transformation. The two fixed points of the transformation are then known. Draw their join, and 7 and 8 are a pair of points on this line harmonic to the Weddle points on the line. Thus the set can be constructed.

Both quartics 2° and 3° contain two constants and have one tangent line fixed, but quartic 3° is invariant under a dihedral $G_{2,3}$. Whether or not the P_8^3 is invariant under a $G_{2,3}$ will decide the question as to with which quadric shall be connected the order of association (123)(456). Since we have just seen that the order of association is possible, one at least of the thirty-six systems of contact cubics is fixed. The quartic, being invariant under a G_3 , must have two other systems of contact cubics fixed, and we can represent them by $\overline{1237}$, $\overline{4568}$ and $\overline{1238}$, $\overline{4567}$. If the P_8^3 is invariant under a $G_{2,3}$ the transformation of period 2 can not be a harmonic perspectivity in a point and plane—else four points in a plane—and is consequently a harmonic perspectivity in two fixed lines. Moreover, it must be of the type $(ab)(cd)(ef)(gh)$ for the P_8^3 , if projective in an order of period 2, is also self-associated in that order, and we saw that the above-mentioned type of period 2 is the only one that exists.

This transformation must send (123)(456) into its inverse, leave $\overline{1237}$, $\overline{4568}$ and $\overline{1238}$, $\overline{4567}$ unaltered; consequently is (14)(26)(35)(78), or (15)(24)(36)(78), or (16)(25)(34)(78). But the P_8^3 is not invariant under any one of these three transformations. Hence we conclude that the order of association (123)(456) is to be connected with quartic 2° and none can be connected with quartic 3° .

Quartic 4° has four fixed undulation tangents, while quartic 5° has none. The orders of association (1234), (1234)(56), and (1234)(56)(78) are excluded since they do not have the same number of fixed lines. The remain-

ing order of period 4 has no fixed lines, hence we connect (1234) (5678) with quartic 5° . The set of points is therefore projective and self-associated in the order (1234) (5678). Let them have the coordinates:

$$\begin{array}{llll} 1: 1, & 1, & 1, & 1, & 2: 1, & i, & -1, & -i, & 3: 1, & -1, & 1, & -1, \\ 4: 1, & -i, & -1, & i, & 5: x_1, & x_2, & x_3, & x_4, & 6: x_1, & ix_2, & -x_3, & -ix_4, \\ & & & & 7: x_1, & -x_2, & x_3, & -x_4, & 8: x_1, & -ix_2, & -x_3, & ix_4, \end{array}$$

The sixteen equations resulting from the identity are reducible to six in $\lambda_5, \lambda_6, \lambda_7, \lambda_8$, namely:

$$\begin{array}{ll} (x_1^2 - x_3^2)(\lambda_5 + \lambda_6 + \lambda_7 + \lambda_8) = 0, & (x_2^2 - x_4^2)(\lambda_5 - \lambda_6 + \lambda_7 - \lambda_8) = 0, \\ (x_1^2 - x_2x_4)(\lambda_5 + \lambda_6 + \lambda_7 + \lambda_8) = 0, & (x_1x_4 - x_2x_3)(\lambda_5 - i\lambda_6 - \lambda_7 + i\lambda_8) = 0, \\ (x_2^2 - x_4^2)(\lambda_5 - \lambda_6 + \lambda_7 - \lambda_8) = 0, & (x_1x_2 - x_3x_4)(\lambda_5 + i\lambda_6 - \lambda_7 - i\lambda_8) = 0. \end{array}$$

For these equations to be consistent, one at least of the multipliers must vanish. If $x_1^2 - x_3^2 = x_1^2 - x_2x_4 = 0$, let $x_1 = 1, x_2 = a$, then $x_3 = \pm 1, x_4 = 1/a$. Using $x_3 = 1$, points 1, 3, 5, 7 lie in a plane, while if $x_3 = -1$, points 1, 3, 6, 8 lie in a plane, so the above hypothesis is impossible. A similar argument shows that if $x_2^2 - x_4^2 = x_2^2 - x_1x_3 = 0$, four points will lie in a plane, so this assumption is likewise untenable.

If $x_1x_4 - x_2x_3 = 0$ by letting $x_1 = 1, x_2 = a, x_3 = b$, then $x_4 = ab$ and the P_8^3 is non-degenerate, lying on a quartic curve. Its coordinates are:

$$\begin{array}{llll} 1: 1, & 1, & 1, & 1, & 2: 1, & i, & -1, & -i, & 3: 1, & -1, & 1, & -1, & 4: 1, & -i, & -1, & i, \\ 5: 1, & a, & b, & ab, & 6: 1, & ia, & -b, & -iab, & 7: 1, & -a, & b, & -ab, & 8: 1, & -ia, & -b, & iab. \end{array}$$

The four lines $\overline{13}, \overline{24}, \overline{57}, \overline{68}$ are generators of a quadric on the curve. The points where these lines cut the line $\overline{e_2e_4}$ of the reference tetrahedron are, respectively, $0, 1, 0, 1; 0, 1, 0, -1; 0, 1, 0, b; 0, 1, 0, -b$. Calling them, respectively, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ the double ratio on $\overline{e_2e_4}$ of $\{\alpha_1\alpha_2\alpha_3\alpha_4\}$ is $\left(\frac{1-b}{1+b}\right)^2$.

The cross generators of the quadric through the points 1, 3, 2, 4 cut the side $\overline{e_3e_4}$ of the reference tetrahedron in four points whose double ratio is -1 . Hence these cross generators are harmonic. The set contains two absolute constants.

If we assume $x_1x_2 - x_3x_4 = 0$, we get a set of points projectively equivalent to the set just discussed, giving no new types. If we assume that more than one of the multipliers of the equations in $\lambda_5, \lambda_6, \lambda_7, \lambda_8$ vanish, then either two points coincide or four points lie in a plane. Hence the order of association (1234) (5678) is connected with quartic 5° , P_8^3 lies on an elliptic quartic and contains two absolute constants.

To quartic 4° we can connect no order of self-association.

Quartic 6° is invariant under a cyclic G_6 whose square $(1, 1, j)$ is connected with the order of association $(123)(456)$ and whose cube $(1, 1, -1)$ with the order $(12)(34)(56)(78)$. In examining the orders of association of period 6 we note that $(123)(45)$ and $(123)(45)(67)$ have not the right number of fixed lines, one. The cube of the order (123456) contains only three cycles of two numbers, while the cube of $(123)(456)(78)$ contains one cycle of two numbers, so neither can be used. The remaining order $(123456)(78)$ is of type E , hence the quartic would not be the general one. Therefore we can not connect an order of self-association with the quartic 6° .

Quartic 7° is the well-known quartic of Klein,* invariant under a G_{168} . With it we connect the order of association (1234567) if it exists. Let us then, with Klein, take the coordinates of the set to be:

$$\begin{array}{llll} 1: 1, & 2, & 2, & 2, \\ 4: 1, & 2\epsilon^{-3}, & 2\epsilon^{-6}, & 2\epsilon^{-5}, \end{array} \quad \begin{array}{llll} 2: 1, & 2\epsilon^{-1}, & 2\epsilon^{-2}, & 2\epsilon^{-4}, \\ 5: 1, & 2\epsilon^{-4}, & 2\epsilon^{-1}, & 2\epsilon^{-2}, \\ 7: 1, & 2\epsilon^{-6}, & 2\epsilon^{-5}, & 2\epsilon^{-3}, \end{array} \quad \begin{array}{llll} 3: 1, & 2\epsilon^{-2}, & 2\epsilon^{-4}, & 2\epsilon^{-1}, \\ 6: 1, & 2\epsilon^{-5}, & 2\epsilon^{-3}, & 2\epsilon^{-6}, \\ 8: 1, & 0, & 0, & 0, \end{array}$$

where $\epsilon^7=1$.

The sixteen equations resulting from substituting these values in the identity are satisfied by choosing $\lambda_1=\lambda_2=\dots=\lambda_7=1$. Thus the set exists, contains no absolute constant and is connected with quartic 7° . With the above coordinates the transformations S and T of Klein are, respectively, (1234567) and $(18)(27)(34)(56)$. Not only is this set self-associated in the identical order and the order (1234567) but in one hundred and sixty-six others. By using S and T we can easily find them and learn they are of four types: forty-eight of type $(178)(246)$, fifty-six of type $(1358)(2674)$, twenty-one of type $(18)(27)(34)(56)$, forty-two of type (1234567) , which, with the identity, make the one hundred and sixty-eight.

The multipliers of the group leaving 8° unaltered are $1, -1, \frac{1+i}{2}$. The square of this is $1, 1, i$; the group leaving 4° unaltered. There is no order of self-association connected with 4° and, consequently, none can be connected with quartic 8° .

No order of association of period nine appears among those of our list, so we have none to connect with quartic 9° .

Quartic 10° is invariant under a G_{12} . The only transformation of period 12 is $(1234)(567)$ which we saw was of type E . Therefore quartic 10° cannot be connected with an order of association.

* Klein, "Elliptischen Modulfunktionen," Bd. I, pp. 701, 724-725.

Hence only four of the ten types of self-projective quartics can be connected with an order of self-association of the P_8^3 , namely:

$$\begin{array}{ll} 1^\circ \text{ with } (12)(34)(56)(78) & 5^\circ \text{ with } (1234)(5678), \\ 2^\circ \text{ with } (123)(456), & 7^\circ \text{ with } (1234567). \end{array}$$

§ 4. If the P_8^3 lies on a rational cubic, it is self-associated in the identical order. Excluding all of type E we find that the P_8^3 on a rational cubic is projective and hence self-associated in the following orders: $(12)(34)(56)$, $(12)(34)(56)(78)$, $(123)(456)$, $(1234)(5678)$, (123456) , (1234567) , and (12345678) . The binary octavics giving the parameters of the P_8^3 for each order of self-association are, respectively:

$$\begin{array}{ll} (x_1^2+x_2^2)(x_1^2+\alpha x_2^2)(x_1^2+\beta x_2^2)x_1x_2=0, & (x_1^4+x_2^4)(x_1^4+\alpha x_2^4)=0, \\ (x_1^2+x_2^2)(x_1^2+\alpha x_1^2)(x_1^2+\beta x_2^2)(x_1^2+\gamma x_2^2)=0, & (x_1^6+x_2^6)x_1x_2=0, \\ (x_1^3+x_2^3)(x_1^3+\alpha x_2^3)x_1x_2=0, & (x_1^7+x_2^7)x_1=0, \quad (x_1^8+x_2^8)=0. \end{array}$$

The number of absolute constants for each P_8^3 is obvious from the above octavics.

In § 2 and § 3 are given the possible types of self-associated sets. A problem that suggests itself for the future is to determine those P_8^3 's which are self-associated in more than one order, and the groups connected with them. This was done for the order (1234567) in § 3, and the one hundred and sixty-eight ways of self-association discussed. We shall do the same now for the P_8^3 on a rational cubic.

In studying the dihedral groups connected with the above P_8^3 we need look for a $G_{2,n}$ only for $n < 5$ if n is odd. If $n \geq 5$, $2n > 8$, and with the odd integer n we could use only n again, which likewise makes a number greater than 8. Hence for n odd $n \leq 3$. Similarly for n even $n \leq 8$. Therefore the possible G_{2n} 's must be found where $n=2, 3, 4, 6, 8$. In the accompanying table are given the numbers for each value of n .

2	3	4	6	8
2	2	2	2	2
2	3	4	6	8
2	3	4	6	8
4	6	8	12	16

The $G_{2,2}$ has two types	$(x_1^2+x_2^2)(x_1^2+\alpha x_2^2)(\alpha x_1^2+x_2^2)x_1x_2,$ $(x_1^4+\alpha x_1^2x_2^2+x_2^4)(x_1^4+\beta x_1^2x_2^2+x_2^4).$
The $G_{2,3}$ has two types	$(x_1^6-x_2^6)x_1x_2, \quad (x_1^6+\alpha x_1^3x_2^3+x_2^6)x_1x_2.$
The $G_{2,4}$ has two types	$(x_1^8-x_2^8), \quad (x_1^8+\alpha x_1^4x_2^4+x_2^8).$
The $G_{2,6}$ has one type	$(x_1^6+x_2^6)x_1x_2.$
The $G_{2,8}$ has one type	$x_1^8+x_2^8.$
The tetrahedral G_4 is	$x_1^8-14x_1^4x_2^4+x_2^8.$

Conclusion.

The general P_8^3 on an elliptic quartic can be self-associated in the following orders: (12), (12)(34), (12)(34)(56), (12)(34)(56)(78), (123)(456)(78), (1234)(5678), (12345)(67), (123456) and (12345678).

The P_8^3 , which is the base points of a net of quadrics, can be self-associated in the following orders, besides the identity: (12)(34)(56)(78), (123)(456), (1234)(5678) and (1234567). To each of these we have connected a particular planar quartic (genus 3), which is self-projective.

The P_8^3 on a rational cubic can be self-associated in the following orders: (12)(34)(56), (12)(34)(56)(78), (123)(456), (1234)(5678), (123456), (1234567) and (12345678). The groups connected with sets are also given. The discussion throughout the paper was restricted to sets of points, of which no two coincide, no three lie on a line, no four in a plane.